Physics 606 Final Exam

Please be well-organized, and show all significant steps clearly in all problems.

You are graded on your work.

An answer, even if correct, will receive zero credit unless it is obtained via the work shown.

Do all your work on blank sheets, and turn them in as scanned files, writing your name clearly.

It is implicit that you pledge to communicate with no one during the exam (except for questions to me about the meaning of the problems).

On an exam there is limited time to evaluate integrals, so:

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c}\,dx = \sqrt{\frac{\pi}{a}}\,e^{\frac{b^2}{4a}+c}$$

$$\int_0^{\infty} re^{-r^2/a^2}\sin(qr)\,dr = \frac{1}{4}\sqrt{\pi}q\,|a|^3 e^{-a^2q^2/4}$$

1. A 1-dimensional harmonic oscillator is perturbed by a constant force F, so that the Hamiltonian is perturbed by

$$V = -Fx$$

where x is the coordinate (which can be written in terms of a and $a^\dagger$ as usual).

Use standard time-independent (stationary-state) perturbation theory in each part below, and give the results in terms of F, m, and ω.

(a) (10) Calculate the first-order shift in the energy of an arbitrary state n.

(b) (15) Calculate the second-order shift in the energy of an arbitrary state n.

2. (25) A charged particle is initially in the ground state of a 1-dimensional harmonic oscillator potential, at $t \to -\infty$. (As usual, the particle has mass m and the potential is $\frac{1}{2}Kx^2 = \frac{1}{2}m\omega^2 x^2$.)

During the period $-\infty < t < \infty$ it is perturbed by an electric field

$$\frac{\mathcal{E}}{\sqrt{\pi}\,\tau} e^{-(t/\tau)^2}$$

so that the perturbing potential (in the Schrodinger picture) is

$$V_t = q\,x\,\frac{\mathcal{E}}{\sqrt{\pi}\,\tau} e^{-(t/\tau)^2} \ .$$

Here q is the charge of the particle and x is the coordinate (which again can be written in terms of a and $a^\dagger$).

Using first-order time-dependent perturbation theory, calculate (to lowest order) the probability that the particle will be excited from the ground state to the first excited state as $t \to \infty$.

3. Squeezed coherent states of the quantized electromagnetic field have current or potential applications in interferometric measurements (e.g. for gravitational wave detection), readout of very weak spectroscopic signals, improving the precision of atomic clocks, calculations for Hawking radiation, and quantum information processing. See the figure on the next page.

The key idea is that one cannot simultaneously reduce the uncertainties in both the amplitude of the field N and its phase ϕ. There is, in fact, an uncertainty relation

$$\Delta N \Delta \phi \geq \text{ constant} \; .$$

In this problem you are to derive this relation, and obtain the constant.

Start with the commutation relation

$$\left[\widehat{a}, \widehat{a}^{\dagger}\right] = 1$$

for the destruction snd creation operators in a specific mode of the electromagnetic field. This relation is exactly the same as that for the harmonic oscillator, with $\widehat{a}$ representing $\widehat{a}_{\overrightarrow{k}\lambda}$ for a particular mode with wavevector $\overrightarrow{k}$ and polarization λ. (The other commutators are zero, just as for the harmonic oscillator.) The number operator is

$$\widehat{N} = \widehat{a}^{\dagger}\widehat{a}$$

with

$$\widehat{a} = e^{i\widehat{\phi}}\sqrt{\widehat{N}}e^{-i\omega t} \; .$$

Note: It is important to be clear and complete in your arguments below.

(a) (10) Obtain the relation

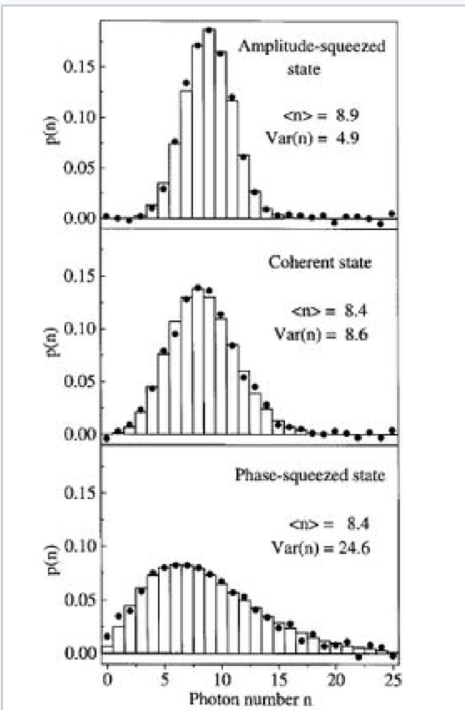
$$e^{i\widehat{\phi}}\widehat{N} - \widehat{N}e^{i\widehat{\phi}} = \text{function of } \widehat{\phi}$$

where you will determine the function.

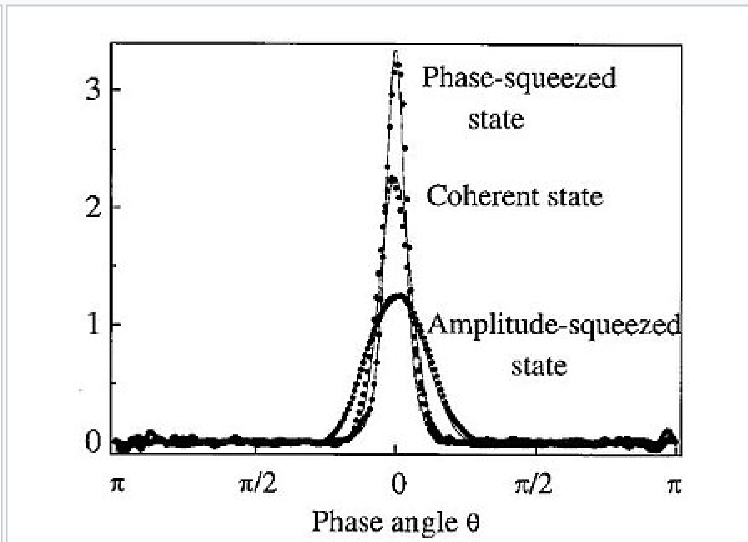
(b) (10) Show that your result in part (a) is satisfied if

$$\left[\widehat{N}, \widehat{\phi}\right] = i \; .$$

(c) (5) Now, recognizing how a result that was derived in class can be generalized, obtain the uncertainty relation above for $\Delta N \Delta \phi$.



Experimental photon number distributions for an amplitude-squeezed state, a coherent state, and a phase squeezed state reconstructed from measurements of the quantum statistics. Bars refer to theory, dots to experimental values.[11]



Pegg-Barnett phase distribution of the three states

Figure 1: Credit: https://en.wikipedia.org/wiki/Squeezed_coherent_state , G. Breitenbach, S. Schiller, and J. Mlynek, "Measurement of the quantum states of squeezed light", Nature, 387, 471 (1997).

4. (a) (13) In the first Born approximation, calculate the differential cross section

$$\frac{d\sigma}{d\Omega}$$

for the Gaussian potential

$$V(r) = V_0 e^{-r^2/a^2} .$$

Give your answer in terms of the momentum transfer q and the various constants.

(This is, of course, a nonrelativistic calculation for a particle of mass m, having kinetic energy $\hbar^2 k^2/2m$, within the standard treatment of scattering theory.)

(b) (12) Calculate the approximate s-wave ($\ell = 0$) phase shift as a function of k at low energy.

Hint: You can start with the fundamental expression for the scattering amplitude in terms of the phase shifts δ_ℓ. Then you can extract the s-wave phase shift δ_0 by using the orthogonality of the Legendre polynomials,

$$\int_{-1}^{1} P_\ell(x)\, P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell\ell'} ,$$

remembering that the momentum transfer is $q = 2k \sin(\theta/2)$.

After evaluating the integral over θ, you should obtain

$$\frac{1}{k} e^{i\delta_0} \sin\delta_0 = \text{function of } k .$$

Finally, by keeping the leading terms in k and δ_0 as $k \to 0$, obtain a solution of the form

$$\delta_0 \to A\, k^n \quad \text{as} \quad k \to 0$$

where you will determine the constants A and n.